Public Capital, Markups, and Labor-Share Dynamics

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Abstract

This paper studies how debt-financed public investment affects labor's share of income when public capital is sector-biased. We build a tractable two-sector general-equilibrium model with monopolistic competition and constant markups. Government issues one-period debt to finance public capital, which augments productivity with higher exposure in the capital-intensive sector. Public capital affects the aggregate labor share through: i) a within-sector factor-price channel operating via the rental–wage ratio and ii) a between-sector market-share channel operating via relative prices and expenditure shares. We derive a closed-form decomposition and sufficient conditions under which both channels reduce the labor share. Calibrated to U.S. annual data (1970–2019), the model's impulse responses show that a debt-financed investment shock raises public capital and real wages, shifts demand toward the capital-intensive sector, and generates a temporary decline in the aggregate labor share.

Keywords: Public capital; labor share; fiscal policy; sectoral reallocation; markups
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1 Introduction

In many countries, the case for public infrastructure is made in plain language. Better roads cut delivery times, reliable power lowers downtime, and broadband turns a remote town into a place where firms can operate. These are the kinds of improvements that voters can see and firms can price. Yet the distributional consequences are harder to read. A new highway may raise output and wages in the short run, but it can also change which sectors expand, which firms gain market share, and who captures the resulting income. This paper studies one specific version of that question: when public capital is financed by debt and it raises productivity, must labor necessarily gain a larger share of income?

This question matters because the labor share has been trending down in many advanced economies, including the United States, for decades (Elsby *et al.*, 2013; Karabarbounis and Neiman, 2014; Dao *et al.*, 2017). A large empirical literature attributes the decline to forces that change factor demands and market structure: cheaper investment goods (Karabarbounis and Neiman, 2014), globalization and offshoring (Elsby *et al.*, 2013), rising concentration and the reallocation of activity toward high-markup, low-labor-share producers (Autor *et al.*, 2020; De Loecker *et al.*, 2020), and measurement issues that matter for cross-country comparisons (Gutiérrez and Piton, 2020). At the same time, policy discussions continue to place heavy weight on public investment as a way to raise productivity and living standards. The core tension is simple: infrastructure is often productive, but labor's claim on the gains is not automatic.

The public-capital literature itself is long and mixed. Early aggregate evidence suggested large productivity effects of infrastructure (Aschauer, 1989). Subsequent work emphasized identification problems and heterogeneity across types of public capital (Gramlich, 1994; Holtz-Eakin, 1994; Pereira, 2000), and industry evidence pointed to plausible channels in which infrastructure is more valuable in vehicle-intensive and logistics-intensive activities (Fernald, 1999). A broad meta-analysis concludes that output elasticities of public capital are positive on average but vary by context and measurement choices (Bom and Ligthart, 2014). In short, public capital can be productive, but its productivity may be sector-specific. That sector specificity is the starting point of this paper.

Our model is deliberately transparent. Time is discrete. A representative household supplies labor inelastically and saves in private capital and one-period government debt (Aschauer, 1989).[1] A local government issues debt and invests in productive public capital. The key policy rule is debt-financed investment: issuance maps one-for-one into public invest-

[1]The full setup is in Section 2. The economy features a representative household, a local government, a final-good producer, and two monopolistically competitive intermediate sectors $j \in \{c, \ell\}$: a capital-intensive sector c and a labor-intensive sector ℓ.

ment, while interest is financed by lump-sum taxation. This delivers a clean mapping from an issuance policy $\{B_{t+1}\}$ to a public-capital path $\{K_t^G\}$, and in steady state $K^G = B/\delta_G$. The production side has two intermediate sectors that differ in (i) capital intensity and (ii) exposure to public capital. Public capital enters sectoral technology via an exposure shifter $g_{j,t} = (K_t^G/\bar{K}^G)^{\psi_j}$ with $\psi_c > \psi_\ell$, so increases in K_t^G reduce marginal costs more in the capital-intensive sector. Each sector features monopolistic competition and constant markups, so the labor share in revenue depends both on the cost-based labor share and on the markup wedge.

The main message is that productive public capital can lower the aggregate labor share even when it raises output and real wages. The reason is that public capital affects the labor share through two channels that are often bundled together in one-sector models. First, there is a within-sector factor-price channel: public capital raises wages in general equilibrium, which changes the rental–wage ratio and hence the cost-minimizing labor share inside each sector. Second, there is a between-sector market-share channel: by lowering the relative price of the capital-intensive composite, public capital shifts expenditure toward that sector when the two sectoral composites are substitutes. If the expanding sector has a lower labor share (because it is more capital intensive and/or has higher markups), the reallocation mechanically pushes down the aggregate labor share. Theorem 1 provides an exact additive decomposition of these two channels.

This perspective connects several empirical literatures that are often discussed separately. The first is the literature on the decline in the labor share and its proximate drivers. Some explanations emphasize factor substitution induced by changes in relative prices (Karabarbounis and Neiman, 2014; Oberfield and Raval, 2021), while others stress reallocations across firms and sectors linked to rising concentration and markups (Autor *et al.*, 2020; De Loecker *et al.*, 2020; Syverson, 2019). A related set of papers highlights that movements in the capital share and profit share are not mirror images, and that housing and rents matter for aggregate factor shares (Rognlie, 2015). Cross-country comparisons require care because corporate sector definitions differ in ways that can bias measured labor shares (Gutiérrez and Piton, 2020). Our framework is not intended to adjudicate among all drivers. Instead, it isolates one mechanism that is directly relevant for fiscal policy: public capital changes both relative factor prices and relative goods prices, so it can generate a decline in the labor share through factor substitution and through reallocation.

The second relevant literature studies the macroeconomic effects of fiscal policy and public investment. Quantitative general equilibrium analyses of fiscal experiments date back at least to (Baxter and King, 1993) and have been complemented by empirical identification strategies based on institutional timing and narratives (Blanchard and Perotti, 2002; Ramey, 2011). State-dependent multipliers are documented in (Auerbach and Gorodnichenko, 2012), and cross-country evidence emphasizes that fiscal effects depend on exchange-rate regimes, openness, and debt levels (Ilzetzki *et al.*, 2013). Infrastructure-specific work highlights the practical challenges posed by implementation lags and anticipation, and provides evidence that infrastructure investment can raise activity over both short and medium horizons (Leduc and Wilson, 2013). The contribution of this paper is to place a distributional object—the labor share—at the center of the analysis using a model in which public capital is productive but sector-asymmetric.

The third literature concerns structural change. Long-run reallocation of activity across broad sectors is a defining feature of modern growth, and it matters for measured factor shares because sectors differ in technologies and markups (Baumol, 1967; Ngai and Pissarides, 2007; Herrendorf *et al.*, 2014). Our two-sector setup is intentionally minimal, but it captures a key implication of this literature: changes in relative prices can generate persistent shifts in sectoral expenditure shares. In our application, public capital moves relative prices endogenously because exposure differs across sectors, and that is what links public investment to labor-share dynamics.

The paper makes three main contributions. First, we model debt-financed public investment with a policy rule that maps issuance into public investment and a law of motion for public capital. This keeps the fiscal block tractable and makes it clear how a temporary issuance shock translates into a hump-shaped public-capital response, both in steady state and in transition.

Second, we derive a closed-form decomposition of the aggregate labor-share response into a within-sector (factor-price) channel and a between-sector (market-share) channel, and we provide sufficient conditions under which both channels are negative.

Third, we calibrate the model using U.S. annual data (1970–2019) and then study impulse responses at quarterly frequency using standard compounding conversions. The simulated responses replicate the model's mechanism: public investment raises public capital and real wages, reduces the relative price of the capital-intensive composite, shifts expenditure toward that sector, and generates a modest but systematic decline in the aggregate labor share.

Section 2 lays out the environment, pricing, and aggregation. Section 3 derives the main decomposition and sufficient conditions. Section 4 presents the calibration and simulation, including impulse responses. Section 5 concludes with policy recommendations provided. The Appendix collects notation, parameter roles, and derivations.

2 Model

2.1 Set up

Time is discrete, $t = 0, 1, 2, \ldots$. There is a representative household, a local government, a competitive final-good producer, and two monopolistically competitive intermediate sectors $j \in \{c, \ell\}$ (capital-intensive c and labor-intensive ℓ). Each intermediate sector consists of a unit mass of differentiated varieties $i \in [0, 1]$.

The representative household has inelastic labor supply $L_t \equiv 1$ and chooses $\{C_t, K_{t+1}, B_{t+1}\}_{t \geq 0}$ to maximize:

$$\max \sum_{t=0}^{\infty} \beta^t \log C_t, \qquad \beta \in (0, 1), \tag{1}$$

subject to the budget constraint

$$C_t + I_t + B_{t+1} + T_t = w_t + r_t K_t + R_t B_t + \Pi_t, \tag{2}$$

and private capital accumulation

$$K_{t+1} = (1 - \delta) K_t + I_t, \qquad \delta \in (0, 1). \tag{3}$$

Here w_t is the real wage, r_t is the real rental rate on private capital, R_t is the gross real return on government debt, T_t is a lump-sum tax, and Π_t denotes aggregate profits rebated to the household.

The household optimality conditions imply

$$\frac{1}{C_t} = \beta \, \frac{1 + r_{t+1} - \delta}{C_{t+1}}, \tag{4}$$

$$\frac{1}{C_t} = \beta \, \frac{R_{t+1}}{C_{t+1}}. \tag{5}$$

Hence, along any deterministic steady state,

$$R = \frac{1}{\beta}, \qquad r = \frac{1}{\beta} - 1 + \delta. \tag{6}$$

The local government issues one-period debt B_{t+1} and invests in productive public capital. Its flow budget constraint is:

$$T_t + B_{t+1} = I_t^G + R_t B_t. \tag{7}$$

Public capital evolves as:

$$K_{t+1}^G = (1-\delta_G)K_t^G + I_t^G, \qquad \delta_G \in (0,1). \tag{8}$$

Policy rule (debt-financed investment) is:

$$I_t^G = B_{t+1}, \qquad T_t = R_t B_t. \tag{9}$$

This rule delivers a transparent mapping from an issuance policy $\{B_{t+1}\}$ to the public-capital path $\{K_t^G\}$, while interest is serviced via lump-sum taxation. Under Equations (9), Equation (7) holds identically.

Along a deterministic steady state with constant $B_{t+1} = B$,

$$I^G = B, \qquad K^G = \frac{B}{\delta_G}. \tag{10}$$

A competitive final-good producer aggregates the two sectoral composites $Y_{c,t}$ and $Y_{\ell,t}$ using a CES technology:

$$Y_t = \left[\phi\, Y_{c,t}^{\frac{\nu-1}{\nu}} + (1-\phi)\, Y_{\ell,t}^{\frac{\nu-1}{\nu}}\right]^{\frac{\nu}{\nu-1}}, \qquad \phi \in (0,1),\ \nu > 0, \tag{11}$$

and takes $(P_{c,t}, P_{\ell,t})$ as given. Let the final good be the numeraire, so its price is normalized to 1.

Cost minimization implies the standard demands and the price index:

$$Y_{c,t} = \phi\, P_{c,t}^{-\nu}\, Y_t, \qquad Y_{\ell,t} = (1-\phi)\, P_{\ell,t}^{-\nu}\, Y_t, \tag{12}$$

$$1 = \left[\phi\, P_{c,t}^{1-\nu} + (1-\phi)\, P_{\ell,t}^{1-\nu}\right]^{\frac{1}{1-\nu}}. \tag{13}$$

Define sectoral revenue (market) shares

$$s_{c,t} \equiv \frac{P_{c,t}\, Y_{c,t}}{Y_t} = \phi\, P_{c,t}^{1-\nu}, \qquad s_{\ell,t} \equiv \frac{P_{\ell,t}\, Y_{\ell,t}}{Y_t} = (1-\phi)\, P_{\ell,t}^{1-\nu}, \qquad s_{c,t} + s_{\ell,t} = 1. \tag{14}$$

2.2 Intermediate sectors, heterogeneity, and public-capital exposure

Within each sector $j \in \{c, \ell\}$, a continuum of varieties $i \in [0,1]$ is aggregated into a sectoral composite

$$Y_{j,t} = \left[\int_0^1 y_{j,t}(i)^{\frac{\theta_j - 1}{\theta_j}}\, di\right]^{\frac{\theta_j}{\theta_j - 1}}, \qquad \theta_j > 1, \tag{15}$$

implying the associated sectoral price index

$$P_{j,t} = \left[\int_0^1 p_{j,t}(i)^{1-\theta_j}\, di\right]^{\frac{1}{1-\theta_j}}. \tag{16}$$

Each variety producer has idiosyncratic productivity $z_{j,t}(i) > 0$ drawn i.i.d. over i and t from a sector-specific distribution F_j with finite $(\theta_j - 1)$-moment:

$$\int z^{\theta_j - 1}\, dF_j(z) < \infty. \tag{17}$$

Public capital affects sectoral technologies with potentially different exposures:

$$g_{j,t} \equiv \left(\frac{K_t^G}{\bar{K}^G}\right)^{\psi_j}, \qquad \psi_j \geq 0, \qquad \psi_c > \psi_\ell, \tag{18}$$

where $\bar{K}^G > 0$ is a scaling constant.

A firm in sector j and variety i produces:

$$y_{j,t}(i) = z_{j,t}(i)\, g_{j,t} \left[\alpha_j K_{j,t}(i)^{\frac{\sigma_j - 1}{\sigma_j}} + (1-\alpha_j) L_{j,t}(i)^{\frac{\sigma_j - 1}{\sigma_j}}\right]^{\frac{\sigma_j}{\sigma_j - 1}}, \qquad \alpha_j \in (0,1),\ \sigma_j > 0,\ \sigma_j \neq 1. \tag{19}$$

The capital-intensive sector satisfies $\alpha_c > \alpha_\ell$. Firms rent capital and hire labor competitively at prices (r_t, w_t).[2]

2.3 Firm problem and aggregation

Fix (r_t, w_t, K_t^G) and suppress time subscripts when no confusion arises. Each firm minimizes cost $(rK + wL)$ subject to Equation (19). Define the unit-cost function associated with the CES aggregator in Equation (19):

$$c_j(r, w) = \left[\alpha_j^{\sigma_j} r^{1-\sigma_j} + (1-\alpha_j)^{\sigma_j} w^{1-\sigma_j}\right]^{\frac{1}{1-\sigma_j}}. \tag{20}$$

Lemma 1 (Firm-level marginal cost)**.** *For any sector j and variety i, marginal cost is*

$$MC_j(i) = \frac{c_j(r, w)}{z_j(i)\, g_j}. \tag{21}$$

[2] For clarity, Table A1 in Appendix A summarizes the model parameters and their roles.

Proof. The production function for variety i in sector j is:

$$y_{j,t}(i) = z_{j,t}(i) g_{j,t} \left[\alpha_j K_{j,t}(i)^{\frac{\sigma_j - 1}{\sigma_j}} + (1-\alpha_j) L_{j,t}(i)^{\frac{\sigma_j-1}{\sigma_j}} \right]^{\frac{\sigma_j}{\sigma_j - 1}}$$

where σ_j is the elasticity of substitution, $\alpha_j \in (0,1)$ is the distribution parameter, $z_{j,t}(i)$ is idiosyncratic productivity, and $g_{j,t}$ is the public capital exposure factor.

For simplicity, let $\rho = \frac{\sigma_j - 1}{\sigma_j}$. The production "kernel" (the bundle of capital and labor) is:

$$Y_{kernel} = \left[\alpha_j K^\rho + (1-\alpha_j) L^\rho \right]^{1/\rho} \tag{22}$$

The total output is $y = AY_{kernel}$, where $A = z_{j,t}(i) g_{j,t}$ represents total factor productivity (TFP). Equation (20) defines the unit-cost function for the kernel.

Step 1: Minimizing cost for the CES kernel: First, we calculate the minimum cost to produce one unit of the kernel Y_{kernel}. We minimize $\mathscr{C} = rK + wL$ subject to:

$$1 = \left[\alpha_j K^\rho + (1-\alpha_j) L^\rho \right]^{1/\rho}$$

Setting up the Lagrangian:

$$\mathscr{L} = rK + wL + \lambda \left(1 - \left[\alpha_j K^\rho + (1-\alpha_j) L^\rho \right]^{1/\rho} \right)$$

The first order conditions (FOCs) are:

$$\frac{\partial \mathscr{L}}{\partial K} = r - \lambda \frac{1}{\rho} [\dots]^{\frac{1}{\rho}-1} \cdot \rho \alpha_j K^{\rho-1} = 0 \implies r = \lambda \alpha_j K^{\rho-1} \tag{23}$$

$$\frac{\partial \mathscr{L}}{\partial L} = w - \lambda \frac{1}{\rho} [\dots]^{\frac{1}{\rho}-1} \cdot \rho (1-\alpha_j) L^{\rho-1} = 0 \implies w = \lambda (1-\alpha_j) L^{\rho-1} \tag{24}$$

Note: Since we normalized output to 1, the term in brackets $[\dots]$ equals 1.

We solve for K and L. From Equation (23):

$$K^{\rho-1} = \frac{r}{\lambda \alpha_j} \implies K = \left(\frac{r}{\lambda \alpha_j} \right)^{\frac{1}{\rho-1}}$$

Recall that $\rho = \frac{\sigma_j - 1}{\sigma_j}$, which implies $\rho - 1 = -\frac{1}{\sigma_j}$. Thus, the exponent becomes $-\sigma_j$:

$$K = \lambda^{\sigma_j} \alpha_j^{\sigma_j} r^{-\sigma_j}$$

By symmetry, for labor:

$$L = \lambda^{\sigma_j}(1-\alpha_j)^{\sigma_j} w^{-\sigma_j}$$

Substitute K and L back into the production constraint $Y_{kernel} = 1$:

$$1 = \left[\alpha_j K^\rho + (1-\alpha_j)L^\rho\right]^{1/\rho}$$
$$1^\rho = \alpha_j \left(\lambda^{\sigma_j}\alpha_j^{\sigma_j} r^{-\sigma_j}\right)^\rho + (1-\alpha_j)\left(\lambda^{\sigma_j}(1-\alpha_j)^{\sigma_j} w^{-\sigma_j}\right)^\rho$$

Using the identity $\rho\sigma_j = \sigma_j - 1$:

$$1 = \lambda^{\sigma_j - 1}\left[\alpha_j^{\sigma_j} r^{1-\sigma_j} + (1-\alpha_j)^{\sigma_j} w^{1-\sigma_j}\right]$$

Solving for λ (which represents the unit cost):

$$\lambda^{1-\sigma_j} = \alpha_j^{\sigma_j} r^{1-\sigma_j} + (1-\alpha_j)^{\sigma_j} w^{1-\sigma_j}$$

$$\lambda = \left[\alpha_j^{\sigma_j} r^{1-\sigma_j} + (1-\alpha_j)^{\sigma_j} w^{1-\sigma_j}\right]^{\frac{1}{1-\sigma_j}}$$

Step 2: Marginal cost with productivity: The firm produces $y_j(i)$ using the technology:

$$y_j(i) = A \cdot \mathscr{F}_j(K, L)$$

where $A = z_j(i)g_j$ is the combined productivity factor.

To produce $y_j(i)$ units of final output, the firm effectively requires $\frac{y_j(i)}{A}$ units of the kernel bundle. The cost of one unit of the bundle is $c_j(r, w)$. Therefore, the total cost (TC) is:

$$TC(y_j(i)) = c_j(r, w) \cdot \frac{y_j(i)}{A} = c_j(r, w)\frac{y_j(i)}{z_j(i)g_j}$$

The marginal cost (MC) is the derivative of total cost with respect to output $y_j(i)$. Since the cost function is linear in $y_j(i)$ (constant returns to scale at the firm level given factor prices), the marginal cost is constant:

$$MC_j(i) = \frac{\partial TC}{\partial y_j(i)} = \frac{c_j(r, w)}{z_j(i)g_j}$$

□

2.4 Price setting and sectoral price index

Variety demand implied by Equation (15) is:

$$y_j(i) = \left(\frac{p_j(i)}{P_j}\right)^{-\theta_j} Y_j. \tag{25}$$

Given $MC_j(i)$, each firm chooses $p_j(i)$ to maximize static profits $\pi_j(i) = \big(p_j(i) - MC_j(i)\big) y_j(i)$.

Lemma 2 (Constant markup pricing)**.** *In sector j, the optimal price is:*

$$p_j(i) = \mu_j\, MC_j(i), \qquad \mu_j \equiv \frac{\theta_j}{\theta_j - 1} > 1. \tag{26}$$

Proof. From Equation (22), the demand faced by a specific variety i in sector j is given by:

$$y_j(i) = \left(\frac{p_j(i)}{P_j}\right)^{-\theta_j} Y_j \tag{27}$$

where $p_j(i)$ is the price of variety i, P_j is the aggregate price index for sector j, Y_j is the aggregate demand for sector j, and $\theta_j > 1$ is the elasticity of substitution between varieties.

The firm operates under monopolistic competition. It treats the aggregate variables P_j and Y_j as exogenous but recognizes that its own sales $y_j(i)$ depend on its chosen price $p_j(i)$.

The firm maximizes static profit $\pi_j(i)$:

$$\pi_j(i) = \left[p_j(i) - MC_j(i)\right] \cdot y_j(i) \tag{28}$$

where $MC_j(i)$ is the marginal cost derived in Lemma 1.

Step 1: Express profit in terms of price: Substitute the demand function (27) into the profit function (28):

$$\pi_j(i) = \left[p_j(i) - MC_j(i)\right]\left(P_j^{\theta_j} Y_j\right) p_j(i)^{-\theta_j}$$

For notational simplicity, let $A = P_j^{\theta_j} Y_j$ (which is constant with respect to the firm's decision). Then:

$$\pi_j(i) = A\left[p_j(i)^{1-\theta_j} - MC_j(i) p_j(i)^{-\theta_j}\right]$$

Step 2: First order condition (FOC): We differentiate the profit function with respect to $p_j(i)$ and set it to zero:

$$\frac{\partial \pi_j(i)}{\partial p_j(i)} = A\left[(1-\theta_j) p_j(i)^{-\theta_j} - MC_j(i)(-\theta_j) p_j(i)^{-\theta_j - 1}\right] = 0$$

Since $A > 0$ and $p_j(i) > 0$, we can divide the entire equation by $A \cdot p_j(i)^{-\theta_j - 1}$:

$$(1 - \theta_j) p_j(i) + \theta_j MC_j(i) = 0$$

Step 3: Solve for optimal price: Rearranging the terms:

$$p_j(i)(\theta_j - 1) = \theta_j MC_j(i)$$

$$p_j(i) = \frac{\theta_j}{\theta_j - 1} MC_j(i)$$

Step 4: The markup definition: Define the markup μ_j as:

$$\mu_j \equiv \frac{\theta_j}{\theta_j - 1}$$

Since $\theta_j > 1$ (varieties are substitutes), it follows that $\mu_j > 1$. The optimal pricing rule becomes:

$$p_j(i) = \mu_j MC_j(i)$$

This matches Equation (26) in Lemma 2 exactly. □

Define the CES productivity aggregator

$$Z_j \equiv \left[\int_0^1 z_j(i)^{\theta_j - 1} \, di \right]^{\frac{1}{\theta_j - 1}}. \tag{29}$$

Lemma 3 (Sectoral price index)**.** *The sectoral price index is:*

$$P_j = \frac{\mu_j \, c_j(r, w)}{Z_j \, g_j}. \tag{30}$$

Proof. From Equation (16), the CES price index for sector j is defined as:

$$P_j = \left[\int_0^1 p_j(i)^{1 - \theta_j} \, di \right]^{\frac{1}{1 - \theta_j}} \tag{31}$$

where $\theta_j > 1$ is the elasticity of substitution.

From Lemma 2 (Equation (26)), the optimal price for variety i is:

$$p_j(i) = \mu_j MC_j(i) = \mu_j \frac{c_j(r, w)}{z_j(i) g_j} \tag{32}$$

where $\mu_j \equiv \frac{\theta_j}{\theta_j - 1}$ is the constant markup, $c_j(r,w)$ is the unit cost of the input bundle, g_j is the public capital exposure, and $z_j(i)$ is the firm-specific productivity.

We defined the sectoral productivity aggregator Z_j in Equation (29).[3]

$$Z_j \equiv \left[\int_0^1 z_j(i)^{\theta_j - 1} di\right]^{\frac{1}{\theta_j - 1}} \tag{33}$$

Step 1: Substitution: Substitute the optimal price (32) into the price index definition (31):

$$P_j = \left[\int_0^1 \left(\frac{\mu_j c_j(r,w)}{g_j z_j(i)}\right)^{1-\theta_j} di\right]^{\frac{1}{1-\theta_j}}$$

Step 2: Factoring out constants: The terms μ_j, $c_j(r,w)$, and g_j are constant across all varieties i. We can pull them out of the integral. Recall that $(AB)^{1-\theta_j} = A^{1-\theta_j} B^{1-\theta_j}$.

$$P_j = \left[\left(\frac{\mu_j c_j(r,w)}{g_j}\right)^{1-\theta_j} \int_0^1 \left(\frac{1}{z_j(i)}\right)^{1-\theta_j} di\right]^{\frac{1}{1-\theta_j}}$$

Apply the outer exponent $\frac{1}{1-\theta_j}$ to the constant term:

$$\left[\left(\frac{\mu_j c_j(r,w)}{g_j}\right)^{1-\theta_j}\right]^{\frac{1}{1-\theta_j}} = \frac{\mu_j c_j(r,w)}{g_j}$$

Now, focus on the integral term. Note that $\left(\frac{1}{z}\right)^{1-\theta_j} = z^{-(1-\theta_j)} = z^{\theta_j - 1}$.

$$\text{Integral Term} = \left[\int_0^1 z_j(i)^{\theta_j - 1} di\right]^{\frac{1}{1-\theta_j}}$$

Step 3: Relating to Z_j: From definition (33), we have:

$$Z_j = \left[\int_0^1 z_j(i)^{\theta_j - 1} di\right]^{\frac{1}{\theta_j - 1}}$$

Raising both sides to the power of -1:

$$Z_j^{-1} = \left[\int_0^1 z_j(i)^{\theta_j - 1} di\right]^{-\frac{1}{\theta_j - 1}}$$

[3]Note: Standard literature (?) defines this aggregator such that it represents the "mean" productivity.

Note that the exponent in our integral term from Step 2 is $\frac{1}{1-\theta_j}$. Since $1-\theta_j = -(\theta_j - 1)$, we have:

$$\frac{1}{1-\theta_j} = -\frac{1}{\theta_j - 1}$$

Thus, the integral term is exactly equal to Z_j^{-1}.

Step 4: Final expression: Combining the constants from Step 2 and the integral result from Step 3:

$$P_j = \frac{\mu_j c_j(r,w)}{g_j} \cdot Z_j^{-1}$$

$$P_j = \frac{\mu_j c_j(r,w)}{Z_j g_j}$$

This matches Equation (30) in Lemma 3 perfectly. □

2.5 Labor income shares

Define the (gross) labor income share in sector j as labor payments divided by sector revenue:

$$LS_j \equiv \frac{w L_j}{P_j Y_j}, \qquad L_j \equiv \int_0^1 L_j(i)\, di. \tag{34}$$

Aggregate labor share is labor payments divided by final output (numeraire price 1):

$$LS \equiv \frac{w(L_c + L_\ell)}{Y} = s_c LS_c + s_\ell LS_\ell, \tag{35}$$

where the second equality uses (14).

2.6 Within-sector factor shares

Let $x \equiv r/w$ denote the rental-wage ratio. Define the labor share in unit cost (not revenue) in sector j:

$$\lambda_j(r,w) \equiv \frac{w\, \partial c_j(r,w)/\partial w}{c_j(r,w)}. \tag{36}$$

Lemma 4 (Closed form for λ_j)**.** *For each sector j,*

$$\lambda_j = \frac{(1-\alpha_j)^{\sigma_j} w^{1-\sigma_j}}{\alpha_j^{\sigma_j} r^{1-\sigma_j} + (1-\alpha_j)^{\sigma_j} w^{1-\sigma_j}} = \left[1 + \left(\frac{\alpha_j}{1-\alpha_j}\right)^{\sigma_j} x^{1-\sigma_j}\right]^{-1}. \tag{37}$$

Proof. From Equation (20), the unit-cost function for sector j is:

$$c_j(r,w) = \left[\alpha_j^{\sigma_j} r^{1-\sigma_j} + (1-\alpha_j)^{\sigma_j} w^{1-\sigma_j}\right]^{\frac{1}{1-\sigma_j}} \tag{38}$$

where r is the rental rate, w is the wage, $\alpha_j \in (0,1)$ is the distribution parameter, and σ_j is the elasticity of substitution.

Equation (36) defines the labor share in unit cost as the elasticity of the cost function with respect to the wage:

$$\lambda_j(r,w) \equiv \frac{w}{c_j(r,w)} \frac{\partial c_j(r,w)}{\partial w} \tag{39}$$

Note: By Shephard's Lemma, $\frac{\partial c_j}{\partial w}$ is the unit labor demand, so $\frac{w}{c_j}\frac{\partial c_j}{\partial w} = \frac{wL_{unit}}{Cost_{unit}}$, which is indeed the cost share.

Step 1: Differentiation: Let D_j be the term inside the brackets of the cost function:

$$D_j \equiv \alpha_j^{\sigma_j} r^{1-\sigma_j} + (1-\alpha_j)^{\sigma_j} w^{1-\sigma_j}$$

So, $c_j = D_j^{\frac{1}{1-\sigma_j}}$.

Now, compute the partial derivative with respect to w:

$$\begin{aligned}\frac{\partial c_j}{\partial w} &= \frac{\partial}{\partial w}\left(D_j^{\frac{1}{1-\sigma_j}}\right) \\ &= \frac{1}{1-\sigma_j} D_j^{\frac{1}{1-\sigma_j}-1} \cdot \frac{\partial D_j}{\partial w}\end{aligned}$$

Differentiating D_j with respect to w:

$$\frac{\partial D_j}{\partial w} = (1-\alpha_j)^{\sigma_j}(1-\sigma_j) w^{-\sigma_j}$$

Substitute this back into the expression for $\frac{\partial c_j}{\partial w}$:

$$\begin{aligned}\frac{\partial c_j}{\partial w} &= \frac{1}{1-\sigma_j} D_j^{\frac{\sigma_j}{1-\sigma_j}} \cdot (1-\alpha_j)^{\sigma_j}(1-\sigma_j) w^{-\sigma_j} \\ &= D_j^{\frac{\sigma_j}{1-\sigma_j}} (1-\alpha_j)^{\sigma_j} w^{-\sigma_j}\end{aligned}$$

Step 2: Calculate elasticity (λ_j): Substitute the result from Step 1 into definition (39):

$$\lambda_j = \frac{w}{c_j}\left(D_j^{\frac{\sigma_j}{1-\sigma_j}} (1-\alpha_j)^{\sigma_j} w^{-\sigma_j}\right)$$

Recall that $c_j = D_j^{\frac{1}{1-\sigma_j}}$.

$$\lambda_j = w \cdot D_j^{-\frac{1}{1-\sigma_j}} \cdot D_j^{\frac{\sigma_j}{1-\sigma_j}} \cdot (1-\alpha_j)^{\sigma_j} \cdot w^{-\sigma_j}$$

Combine terms:

- **Wage terms:** $w^1 \cdot w^{-\sigma_j} = w^{1-\sigma_j}$
- D_j **terms:** $D_j^{\frac{\sigma_j - 1}{1-\sigma_j}} = D_j^{-1} = \frac{1}{D_j}$

Thus:

$$\lambda_j = \frac{(1-\alpha_j)^{\sigma_j} w^{1-\sigma_j}}{D_j} = \frac{(1-\alpha_j)^{\sigma_j} w^{1-\sigma_j}}{\alpha_j^{\sigma_j} r^{1-\sigma_j} + (1-\alpha_j)^{\sigma_j} w^{1-\sigma_j}} \tag{40}$$

This matches the first equality in Equation (37) of Lemma 4.

Step 3: Express in terms of relative factor prices (x): Let $x \equiv r/w$. We want to rearrange Equation (40). Divide the numerator and the denominator by the numerator term $(1-\alpha_j)^{\sigma_j} w^{1-\sigma_j}$:

$$\lambda_j = \frac{1}{\frac{\alpha_j^{\sigma_j} r^{1-\sigma_j}}{(1-\alpha_j)^{\sigma_j} w^{1-\sigma_j}} + 1}$$

Group the terms with exponents:

$$\frac{\alpha_j^{\sigma_j} r^{1-\sigma_j}}{(1-\alpha_j)^{\sigma_j} w^{1-\sigma_j}} = \left(\frac{\alpha_j}{1-\alpha_j}\right)^{\sigma_j} \left(\frac{r}{w}\right)^{1-\sigma_j}$$

Substitute $x = r/w$:

$$\lambda_j = \frac{1}{1 + \left(\frac{\alpha_j}{1-\alpha_j}\right)^{\sigma_j} x^{1-\sigma_j}}$$

Or equivalently:

$$\lambda_j = \left[1 + \left(\frac{\alpha_j}{1-\alpha_j}\right)^{\sigma_j} x^{1-\sigma_j}\right]^{-1}$$

This perfectly matches the second equality in Equation (37). □

Proposition 1 (Sectoral labor share in revenue)**.** *For each sector j,*

$$LS_j = \frac{1}{\mu_j}\lambda_j = \frac{1}{\mu_j}\left[1 + \left(\frac{\alpha_j}{1-\alpha_j}\right)^{\sigma_j} x^{1-\sigma_j}\right]^{-1}. \tag{41}$$

Proof. Equation (34) defines the labor share in sector j revenue as:

$$LS_j \equiv \frac{w L_j}{P_j Y_j} \tag{42}$$

where wL_j is total labor compensation in the sector and P_jY_j is total sectoral revenue.

From Lemma 2, firms set prices as a constant markup over marginal cost:

$$p_j(i) = \mu_j MC_j(i) \tag{43}$$

From Lemma 4, the share of labor in the unit cost is:

$$\lambda_j = \left[1 + \left(\frac{\alpha_j}{1-\alpha_j}\right)^{\sigma_j} x^{1-\sigma_j}\right]^{-1} \tag{44}$$

Step 1: Relate total cost to revenue: First, we analyze the relationship between a firm's total variable cost and its revenue. Because the technology exhibits constant returns to scale (CRS) in variable inputs (capital and labor), the marginal cost (MC) is equal to the average variable cost (AVC).

$$\text{Total Cost}_j(i) = MC_j(i) \cdot y_j(i)$$

From the pricing rule (43), we know that $MC_j(i) = \frac{p_j(i)}{\mu_j}$. Substitute this into the total cost equation:

$$\text{Total Cost}_j(i) = \frac{p_j(i)}{\mu_j} \cdot y_j(i) = \frac{1}{\mu_j}\left(p_j(i)y_j(i)\right)$$

Since this holds for every variety i, we can aggregate over the sector (integral from 0 to 1):

$$\text{Sector Total Cost}_j = \int_0^1 TotalCost_j(i)di = \frac{1}{\mu_j}\int_0^1 p_j(i)y_j(i)di$$

$$\text{Sector Total Cost}_j = \frac{1}{\mu_j}P_jY_j$$

Step 2: Relate labor cost to total cost: By Shephard's Lemma (or the property of the cost function derived in Lemma 4), the optimal payment to labor is a fraction λ_j of the total cost.

$$wL_j = \lambda_j \times (\text{Sector Total Cost}_j)$$

Substitute the result from Step 1:

$$wL_j = \lambda_j\left(\frac{1}{\mu_j}P_jY_j\right) \tag{45}$$

Step 3: Solve for labor share in revenue: Substitute Equation (45) into the definition of LS_j (42):

$$LS_j = \frac{wL_j}{P_j Y_j} = \frac{\lambda_j \frac{1}{\mu_j} P_j Y_j}{P_j Y_j}$$

Canceling $P_j Y_j$:

$$LS_j = \frac{\lambda_j}{\mu_j}$$

Step 4: Final expression: Finally, substitute the closed-form expression for λ_j from Equation (44):

$$LS_j = \frac{1}{\mu_j}\left[1 + \left(\frac{\alpha_j}{1-\alpha_j}\right)^{\sigma_j} x^{1-\sigma_j}\right]^{-1}$$

This matches Equation (41) in Proposition 1 exactly. □

Corollary 1 (Ranking of sectoral labor shares)**.** *Fix* $x > 0$. *If* $\alpha_c > \alpha_\ell$ *and* $\mu_c \geq \mu_\ell$ *and* $\sigma_c = \sigma_\ell = \sigma$*, then* $LS_c < LS_\ell$.

Proof. Assumptions: i) *Capital intensity:* Sector c is more capital intensive than sector l, meaning the distribution parameter for capital satisfies $\alpha_c > \alpha_l$. Both are in $(0,1)$; ii) *Markups:* The markup in the capital-intensive sector is at least as high as in the labor-intensive sector: $\mu_c \geq \mu_l > 1$; iii) *Elasticities:* The elasticity of substitution is identical across sectors: $\sigma_c = \sigma_l = \sigma > 0$; iv) *Factor prices:* The relative rental-wage ratio $x \equiv r/w$ is positive $(x > 0)$.

From Proposition 1, the labor share in sector j is given by:

$$LS_j = \frac{1}{\mu_j}\left[1 + \left(\frac{\alpha_j}{1-\alpha_j}\right)^{\sigma_j} x^{1-\sigma_j}\right]^{-1} \tag{46}$$

Step 1: Define the auxiliary function: Since $\sigma_c = \sigma_l = \sigma$, we can define a function $f(\alpha)$ representing the term inside the brackets that depends on α:

$$A(\alpha) \equiv \left(\frac{\alpha}{1-\alpha}\right)^{\sigma} x^{1-\sigma} \tag{47}$$

The labor share in cost (λ_j) can then be written as a function of α:

$$\lambda(\alpha) = \frac{1}{1 + A(\alpha)} \tag{48}$$

Step 2: Analyze monotonicity: First, consider the term $\frac{\alpha}{1-\alpha}$. Let $g(\alpha) = \frac{\alpha}{1-\alpha}$. Differentiating with respect to α:

$$g'(\alpha) = \frac{1(1-\alpha) - \alpha(-1)}{(1-\alpha)^2} = \frac{1-\alpha+\alpha}{(1-\alpha)^2} = \frac{1}{(1-\alpha)^2} > 0$$

Thus, the ratio $\frac{\alpha}{1-\alpha}$ is strictly increasing in α for $\alpha \in (0,1)$.

Next, consider $A(\alpha) = [g(\alpha)]^{\sigma} x^{1-\sigma}$. Since $\sigma > 0$, $[g(\alpha)]^{\sigma}$ is strictly increasing in α. Since $x > 0$, $x^{1-\sigma}$ is a positive constant. Therefore, $A(\alpha)$ is strictly increasing in α.

Finally, consider $\lambda(\alpha) = [1 + A(\alpha)]^{-1}$. Since $A(\alpha)$ is strictly increasing, the denominator $1 + A(\alpha)$ is strictly increasing. Consequently, $\lambda(\alpha)$ is strictly decreasing in α.

Step 3: Compare sectors: Given $\alpha_c > \alpha_l$, the result from Step 2 implies:

$$\lambda_c = \lambda(\alpha_c) < \lambda(\alpha_l) = \lambda_l \tag{49}$$

This means the labor share in production cost is lower for the capital-intensive sector.

Now, consider the labor share in revenue, $LS_j = \frac{\lambda_j}{\mu_j}$. We are given $\mu_c \geq \mu_l$. This implies:

$$\frac{1}{\mu_c} \leq \frac{1}{\mu_l} \tag{50}$$

Combining the inequalities:

$$\begin{aligned} LS_c &= \frac{1}{\mu_c}\lambda_c \\ &< \frac{1}{\mu_c}\lambda_l \quad (\text{since } \lambda_c < \lambda_l) \\ &\leq \frac{1}{\mu_l}\lambda_l \quad (\text{since } \frac{1}{\mu_c} \leq \frac{1}{\mu_l}) \\ &= LS_l \end{aligned}$$

Thus, $LS_c < LS_l$. □

3 Debt-financed public capital and the labor share

This section states results in steady state, using Equation (6) and Equation (10). All comparative statics below can be read as responses to a permanent change in B.

3.1 Relative prices and market shares

From Equations (30) and (18), the relative sectoral price is:

$$\mathscr{R} \equiv \frac{P_c}{P_\ell} = \frac{\mu_c}{\mu_\ell} \cdot \frac{c_c(r,w)}{c_\ell(r,w)} \cdot \frac{Z_\ell}{Z_c} \cdot \left(\frac{K^G}{\bar{K}^G}\right)^{\psi_\ell - \psi_c}. \tag{51}$$

Lemma 5 (Market share response to relative prices)**.** *The revenue share s_c satisfies*

$$\frac{d\,s_c}{d\log\mathscr{R}} = (1-\nu)\,s_c(1-s_c). \tag{52}$$

Proof. Step 1: Definitions and setup: From the model setup, the final good is produced using a CES aggregator with elasticity of substitution $\nu > 0$ and distribution parameter $\phi \in (0,1)$. The sectoral revenue shares s_c and s_l are defined as:

$$s_c \equiv \frac{P_c\,Y_c}{Y} = \phi\,P_c^{1-\nu} \tag{53}$$

$$s_l \equiv \frac{P_l\,Y_l}{Y} = (1-\phi)P_l^{1-\nu} \tag{54}$$

Since the final good is the numeraire ($P = 1$), the ideal price index condition implies:

$$1 = \phi\,P_c^{1-\nu} + (1-\phi)P_l^{1-\nu} \implies s_c + s_l = 1 \tag{55}$$

Let $\mathscr{R} \equiv P_c/P_l$ denote the relative price.

Step 2: Expressing the share ratio: We start by taking the ratio of the two shares. Using Equations (53) and (54):

$$\frac{s_c}{s_l} = \frac{\phi\,P_c^{1-\nu}}{(1-\phi)P_l^{1-\nu}} = \frac{\phi}{1-\phi}\left(\frac{P_c}{P_l}\right)^{1-\nu} \tag{56}$$

Substituting $s_l = 1-s_c$ and $\mathscr{R} = P_c/P_l$:

$$\frac{s_c}{1-s_c} = \frac{\phi}{1-\phi}\mathscr{R}^{1-\nu} \tag{57}$$

Step 3: Log-differentiation: Take the natural logarithm (ln) of both sides of Equation (57):

$$\ln\left(\frac{s_c}{1-s_c}\right) = \ln\left(\frac{s_c}{s_l}\right) = \ln\left(\frac{\phi}{1-\phi}\right) + (1-\nu)\ln(\mathscr{R}) \tag{58}$$

Expand the left-hand side:

$$\ln(s_c) - \ln(1-s_c) = \text{constant} + (1-\nu)\ln(\mathscr{R}) \tag{59}$$

Now, differentiate with respect to $\ln(\mathscr{R})$. Let $d(\cdot)$ denote the differential operator.

$$d[\ln(s_c)] - d[\ln(1-s_c)] = (1-\nu)d[\ln(\mathscr{R})] \tag{60}$$

Using the chain rule $\frac{d}{dx}\ln(f(x)) = \frac{f'(x)}{f(x)}$:

$$\frac{1}{s_c} d s_c - \frac{1}{1-s_c} d(1-s_c) = (1-\nu) d \ln \mathscr{R} \tag{61}$$

Note that $d(1-s_c) = -d s_c$. Substituting this back:

$$\frac{d s_c}{s_c} - \frac{-d s_c}{1-s_c} = (1-\nu) d \ln \mathscr{R} \tag{62}$$

$$d s_c \left(\frac{1}{s_c} + \frac{1}{1-s_c} \right) = (1-\nu) d \ln \mathscr{R} \tag{63}$$

Step 4: Algebraic simplification: Combine the terms in the parentheses:

$$\frac{1}{s_c} + \frac{1}{1-s_c} = \frac{(1-s_c)+s_c}{s_c(1-s_c)} = \frac{1}{s_c(1-s_c)} \tag{64}$$

Substitute this back into the differential equation:

$$d s_c \cdot \frac{1}{s_c(1-s_c)} = (1-\nu) d \ln \mathscr{R} \tag{65}$$

Multiply both sides by $s_c(1-s_c)$:

$$d s_c = (1-\nu) s_c (1-s_c) d \ln \mathscr{R} \tag{66}$$

Finally, rearrange to find the derivative with respect to the log relative price:

$$\frac{d s_c}{d \ln \mathscr{R}} = (1-\nu) s_c (1-s_c) \tag{67}$$

□

3.2 General-equilibrium wage feedback

With the final-good price normalized to 1, Equation (13) implies a restriction on (w, K^G). Let $\bar{\psi} \equiv s_c \psi_c + (1-s_c)\psi_\ell$ and $\bar{\lambda} \equiv s_c \lambda_c + (1-s_c)\lambda_\ell$.

Proposition 2 (Wage response to public capital)**.** *Holding r fixed, a marginal change in K^G changes the real wage according to*

$$\frac{d \log w}{d \log K^G} = \frac{\bar{\psi}}{\bar{\lambda}}, \qquad \bar{\psi} > 0,\ \bar{\lambda} \in (0,1). \tag{68}$$

Proof. Step 1: Setup and definitions: We are analyzing the response of the real wage w to a change in public capital K^G, holding the rental rate r fixed.

- **Sectoral price Equation (Lemma 3):**

$$P_j = \frac{\mu_j c_j(r, w)}{Z_j g_j} \tag{69}$$

where $g_j = (K^G/\overline{K}^G)^{\psi_j}$.

- **Numeraire condition:** The final good is the numeraire, so its ideal price index equals 1:

$$1 = [\phi P_c^{1-\nu} + (1-\phi) P_l^{1-\nu}]^{\frac{1}{1-\nu}} \tag{70}$$

- **Factor shares (λ_j):** The share of labor in unit cost is defined as $\lambda_j \equiv \frac{w}{c_j}\frac{\partial c_j}{\partial w}$.

Step 2: Log-differentiation of sectoral prices: Take the natural logarithm of the sectoral price equation:

$$\ln P_j = \ln \mu_j + \ln c_j(r, w) - \ln Z_j - \ln g_j \tag{71}$$

Substitute $\ln g_j = \psi_j \ln K^G - \psi_j \ln \overline{K}^G$:

$$\ln P_j = \text{const} + \ln c_j(r, w) - \psi_j \ln K^G \tag{72}$$

Differentiate this expression with respect to K^G and w, holding r constant ($dr = 0$):

$$d \ln P_j = d \ln c_j(r, w) - \psi_j d \ln K^G \tag{73}$$

Using the definition of the labor share λ_j, the change in unit cost is:

$$d \ln c_j = \underbrace{\frac{\partial \ln c_j}{\partial \ln r} d \ln r}_{0 \text{ (fixed } r)} + \frac{\partial \ln c_j}{\partial \ln w} d \ln w = \lambda_j d \ln w \tag{74}$$

Substituting this back:

$$d \ln P_j = \lambda_j d \ln w - \psi_j d \ln K^G \tag{75}$$

Step 3: Log-differentiation of the aggregate price index: Consider the numeraire condition $P(P_c, P_l) = 1$. Totally differentiate the logarithm of the price index:

$$d \ln P = \sum_{j \in \{c,l\}} \frac{\partial \ln P}{\partial \ln P_j} d \ln P_j = 0 \tag{76}$$

For a CES price index, the elasticity of the index with respect to a sectoral price is exactly the sectoral revenue share s_j:

$$\frac{\partial \ln P}{\partial \ln P_j} = s_j \tag{77}$$

Thus, the condition $d \ln P = 0$ implies:

$$s_c d \ln P_c + s_l d \ln P_l = 0 \tag{78}$$

Using the fact that $s_l = 1 - s_c$:

$$s_c d \ln P_c + (1 - s_c) d \ln P_l = 0 \tag{79}$$

4. Solving for the wage response: Substitute the expression for $d \ln P_j$ from Equation (75) into Equation (78):

$$s_c(\lambda_c d \ln w - \psi_c d \ln K^G) + (1 - s_c)(\lambda_l d \ln w - \psi_l d \ln K^G) = 0 \tag{80}$$

Group the terms associated with $d \ln w$ and $d \ln K^G$:

$$[s_c \lambda_c + (1 - s_c)\lambda_l] d \ln w - [s_c \psi_c + (1 - s_c)\psi_l] d \ln K^G = 0 \tag{81}$$

Using the definitions $\overline{\lambda} \equiv s_c \lambda_c + (1 - s_c)\lambda_l$ and $\overline{\psi} \equiv s_c \psi_c + (1 - s_c)\psi_l$:

$$\overline{\lambda} d \ln w - \overline{\psi} d \ln K^G = 0 \tag{82}$$

Rearranging to solve for the elasticity:

$$\overline{\lambda} d \ln w = \overline{\psi} d \ln K^G \tag{83}$$

$$\frac{d \ln w}{d \ln K^G} = \frac{\overline{\psi}}{\overline{\lambda}} \tag{84}$$

□

Since $x = r/w$ and r is pinned in steady state by (6), Proposition 2 implies

$$\frac{d \log x}{d \log K^G} = -\frac{\bar{\psi}}{\bar{\lambda}}. \tag{85}$$

3.3 Main result: labor share decomposition

Define, for each sector j, the sensitivity of λ_j to x:[4]

$$\frac{d\lambda_j}{d\log x} = -(1-\sigma_j)\lambda_j(1-\lambda_j). \tag{86}$$

Theorem 1 (Effect of public capital on aggregate labor share)**.** *Holding r fixed, the steady-state response of the aggregate labor share to public capital satisfies*

$$\frac{dLS}{d\log K^G} = \underbrace{\sum_{j\in\{c,\ell\}} s_j\left(\frac{1}{\mu_j}\frac{d\lambda_j}{d\log x}\right)\frac{d\log x}{d\log K^G}}_{\textit{within-sector (factor-price) channel}} + \underbrace{(LS_c - LS_\ell)\frac{ds_c}{d\log K^G}}_{\textit{between-sector (market-share) channel}}, \tag{87}$$

where

$$\frac{d\log x}{d\log K^G} = -\frac{\bar{\psi}}{\bar{\lambda}}, \tag{88}$$

$$\frac{ds_c}{d\log K^G} = (1-\nu)s_c(1-s_c)\frac{d\log\mathscr{R}}{d\log K^G}, \tag{89}$$

$$\frac{d\log\mathscr{R}}{d\log K^G} = (\lambda_c-\lambda_\ell)\frac{d\log w}{d\log K^G} - (\psi_c-\psi_\ell) = (\lambda_c-\lambda_\ell)\frac{\bar{\psi}}{\bar{\lambda}} - (\psi_c-\psi_\ell). \tag{90}$$

Proof. Step 1: Setup and definitions: The aggregate labor share (LS) is the weighted average of sectoral labor shares (LS_j) weighted by their revenue shares (s_j):

$$LS = s_c LS_c + s_l LS_l \tag{91}$$

where $s_l = 1 - s_c$. The sectoral labor share is related to the labor share in cost (λ_j) by the markup (μ_j):

$$LS_j = \frac{\lambda_j}{\mu_j} \tag{92}$$

The relative rental-wage ratio is defined as $x \equiv r/w$. Since r is fixed in the steady state, $d\ln x = -d\ln w$.

Step 2: Total differentiation of aggregate labor share: We totally differentiate the expression for

[4] This identity follows by differentiating Equation (37) directly.

LS with respect to $\log K^G$.

$$d\,LS = d(s_c LS_c + (1-s_c)LS_l) \tag{93}$$
$$= s_c\,d\,LS_c + LS_c\,d\,s_c + (1-s_c)d\,LS_l - LS_l\,d\,s_c \tag{94}$$
$$= [s_c\,d\,LS_c + (1-s_c)d\,LS_l] + (LS_c - LS_l)d\,s_c \tag{95}$$

Dividing by $d\log K^G$:

$$\frac{d\,LS}{d\log K^G} = \underbrace{\sum_j s_j \frac{d\,LS_j}{d\log K^G}}_{\text{Within-Sector}} + \underbrace{(LS_c - LS_l)\frac{d\,s_c}{d\log K^G}}_{\text{Between-Sector}} \tag{96}$$

Step 3: Analyzing the within-sector channel: Recall that $LS_j = \mu_j^{-1}\lambda_j$. Since markups μ_j are constant:

$$\frac{d\,LS_j}{d\log K^G} = \frac{1}{\mu_j}\frac{d\lambda_j}{d\log K^G} \tag{97}$$

Using the chain rule with x:

$$\frac{d\lambda_j}{d\log K^G} = \frac{d\lambda_j}{d\log x}\frac{d\log x}{d\log K^G} \tag{98}$$

Substituting this back into the first term of Equation (96):

$$\text{Within-Sector Term} = \sum_{j\in\{c,l\}} s_j \left(\frac{1}{\mu_j}\frac{d\lambda_j}{d\log x}\right)\frac{d\log x}{d\log K^G} \tag{99}$$

From Proposition 2, we know $\frac{d\ln w}{d\ln K^G} = \frac{\overline{\psi}}{\overline{\lambda}}$. Since r is fixed, $\frac{d\ln x}{d\ln K^G} = -\frac{\overline{\psi}}{\overline{\lambda}}$.

Step 4: Analyzing the between-sector channel: We need to determine $\frac{d\,s_c}{d\log K^G}$. Using the chain rule and Lemma 5:

$$\frac{d\,s_c}{d\log K^G} = \frac{d\,s_c}{d\log\mathscr{R}}\frac{d\log\mathscr{R}}{d\log K^G} \tag{100}$$
$$= (1-\nu)s_c(1-s_c)\frac{d\log\mathscr{R}}{d\log K^G} \tag{101}$$

Now we find the change in relative price $\mathscr{R} = P_c/P_l$.

$$\ln\mathscr{R} = \ln P_c - \ln P_l \tag{102}$$

From the proof of Proposition 2, $d\ln P_j = \lambda_j d\ln w - \psi_j d\ln K^G$. Therefore:

$$d\ln\mathscr{R} = (\lambda_c d\ln w - \psi_c d\ln K^G) - (\lambda_l d\ln w - \psi_l d\ln K^G) \tag{103}$$

$$= (\lambda_c - \lambda_l) d\ln w - (\psi_c - \psi_l) d\ln K^G \tag{104}$$

Dividing by $d\ln K^G$ and substituting $\frac{d\ln w}{d\ln K^G} = \frac{\overline{\psi}}{\overline{\lambda}}$:

$$\frac{d\log\mathscr{R}}{d\log K^G} = (\lambda_c - \lambda_l)\frac{\overline{\psi}}{\overline{\lambda}} - (\psi_c - \psi_l) \tag{105}$$

Step 5: Final assembly: Substituting the components from Sections 3 and 4 back into Equation (96) yields the exact expression in Theorem 1:

$$\frac{dLS}{d\log K^G} = \sum_j s_j \left(\frac{1}{\mu_j}\frac{d\lambda_j}{d\log x}\right)\frac{d\log x}{d\log K^G} + (LS_c - LS_l)\frac{ds_c}{d\log K^G} \tag{106}$$

$\square$

Theorem 1 provides a clean additive decomposition. The first term captures how public capital changes relative factor prices (x), which induces firms to substitute between labor and capital (governed by σ_j). The second term captures how public capital changes relative sectoral prices ($\mathscr{R}$), which reallocates market share (s_c) between sectors with different labor intensities (governed by ν).

Corollary 2 (A sufficient condition for debt-financed public capital to reduce LS)**.** *Assume $\nu > 1$, $\psi_c > \psi_\ell$, and $\sigma_j > 1$ for both j. If $LS_c < LS_\ell$ and*

$$(\psi_c - \psi_\ell) > (\lambda_c - \lambda_\ell)\frac{\bar{\psi}}{\bar{\lambda}}, \tag{107}$$

then $\frac{ds_c}{d\log K^G} > 0$ and the market-share channel in (87) is strictly negative. Moreover, the within-sector channel is also strictly negative, so $\frac{dLS}{d\log K^G} < 0$. Under the policy rule (9)–(10), this implies $\frac{dLS}{d\log B} < 0$ in steady state.

Proof. Step 1: Assumptions: We adopt the following parameter restrictions and conditions from the Corollary statement: i) Substitutability: $\nu > 1$ (sectoral goods are substitutes) and $\sigma_j > 1$ (capital and labor are substitutes in both sectors); ii) Sectoral characteristics: Sector c is capital-intensive relative to sector l, implying $LS_c < LS_l$; iii) Public capital exposure: Sector

c benefits more from public capital, so $\psi_c > \psi_l$; iv) Sufficient Condition (107):

$$(\psi_c - \psi_l) > (\lambda_c - \lambda_l)\frac{\overline{\psi}}{\overline{\overline{\lambda}}} \tag{108}$$

Our goal is to determine the sign of $\frac{dLS}{d\log K^G}$.

Step 2: Analyzing the within-sector channel: From Theorem 1, the within-sector component is:

$$\text{Within} = \sum_j s_j \left(\frac{1}{\mu_j}\frac{d\lambda_j}{d\log x}\right)\frac{d\log x}{d\log K^G} \tag{109}$$

Step 2a: Sign of $d\lambda_j/d\log x$: Differentiating the labor share in cost λ_j:

$$\frac{d\lambda_j}{d\log x} = (\sigma_j - 1)\lambda_j(1-\lambda_j) \tag{110}$$

Since $\sigma_j > 1$ and $\lambda_j \in (0,1)$, this derivative is strictly positive. An increase in the rental-wage ratio ($x = r/w$) increases the labor share in cost because firms substitute away from capital strongly enough to overcome the price change.

Step 2b: Sign of $d\log x/d\log K^G$: From Proposition 2, keeping r fixed implies $d\log x = -d\log w$. Thus:

$$\frac{d\log x}{d\log K^G} = -\frac{\overline{\psi}}{\overline{\overline{\lambda}}} \tag{111}$$

Since $\overline{\psi}, \overline{\lambda} > 0$, this term is strictly negative.

Conclusion for within-sector channel: The product of a positive term and a negative term is negative.

$$\text{Within-sector effect} < 0 \tag{112}$$

3. Analyzing the between-sector channel: From Theorem 1, the between-sector (market-share) component is:

$$\text{Between} = (LS_c - LS_l)\frac{ds_c}{d\log K^G} \tag{113}$$

We expand the change in market share using Lemma 5:

$$\frac{ds_c}{d\log K^G} = (1-\nu)s_c(1-s_c)\frac{d\log \mathscr{R}}{d\log K^G} \tag{114}$$

Step 3a: Sign of relative price change: From Theorem 1 (Equation (90)), the relative price response is:

$$\frac{d\log \mathscr{R}}{d\log K^G} = (\lambda_c - \lambda_l)\frac{\overline{\psi}}{\overline{\overline{\lambda}}} - (\psi_c - \psi_l) \tag{115}$$

Rearranging the sufficient condition (108):

$$(\lambda_c - \lambda_l)\frac{\overline{\psi}}{\overline{\lambda}} - (\psi_c - \psi_l) < 0 \tag{116}$$

Thus, the relative price of the capital-intensive good decreases: $\frac{d\log \mathscr{R}}{d\log K^G} < 0$.
Step 3b: Sign of market share change: Since $\nu > 1$, the term $(1-\nu)$ is negative. Combining signs:

$$\frac{d s_c}{d\log K^G} = (\text{negative}) \times (\text{positive}) \times (\text{negative}) > 0 \tag{117}$$

The market share of sector c increases.
Step 3c: Sign of between-sector effect: We assumed sector c is capital intensive, so $LS_c < LS_l$, which implies $(LS_c - LS_l) < 0$.

$$\text{Between} = (\text{negative}) \times (\text{positive}) < 0 \tag{118}$$

Step 4: Total effect: Combining the two channels:

$$\frac{d\,LS}{d\log K^G} = \underbrace{\text{Within}}_{(-)} + \underbrace{\text{Between}}_{(-)} < 0 \tag{119}$$

Since steady-state public capital is proportional to debt ($K^G = B/\delta_G$), an increase in debt increases K^G, which in turn unambiguously reduces the aggregate labor share under these conditions. □

4 Calibration and Simulation

This section calibrates the model and conducts a data-based simulation using U.S. annual data. The sample is 1970–2019 to align the labor-share series with the public-capital stock series.

4.1 Data

Series and construction Let Y_t denote nominal GDP. Government investment is measured by gross government investment, and the public-capital stock is proxied by the current-cost net stock of government fixed assets. Government depreciation is measured by government consumption of fixed capital. The aggregate labor share is measured by the Penn World Table labor compensation share in GDP.

To map the model's two sectors into observable aggregates, the labor-intensive sector is identified with services and the capital-intensive sector with non-services (goods plus structures). The capital-intensive revenue share is constructed as:

$$s_{c,t}^{\text{data}} \equiv 1 - \frac{Y_t^{\text{services}}}{Y_t},$$

where Y_t^{services} is GDP: services.

The public-capital depreciation rate is estimated as

$$\delta_G^{\text{data}} \equiv \frac{\text{CFC}_t^G}{K_t^G},$$

where CFC_t^G is government consumption of fixed capital and K_t^G is the government net stock of fixed assets. We use the sample mean $\bar{\delta}_G$ for calibration.

The key sample means used as calibration targets are: $\bar{s}_c = 0.442, \overline{K^G/Y} = 0.734, \overline{LS} = 0.614, \bar{\delta}_G = 0.0426$. For reference, gross government investment averages $\overline{I^G/Y} = 0.043$ over the same sample.

All macro series are downloaded from FRED. Underlying sources are: BEA NIPA (GDP, services GDP, gross government investment, and government consumption of fixed capital), BEA Fixed Assets (government net stock of fixed assets), and Penn World Table (labor share).[5]

4.2 Frequency and timing

For impulse-response analysis, however, we additionally solve the model at a quarterly frequency. This is standard in the DSGE literature and yields smoother transition dynamics without changing the underlying identification of targets.

Annual calibration versus quarterly IRFs Table 1 reports parameters in annual units. When producing quarterly IRFs, we convert time-preference and depreciation rates using exact compounding:

$$\beta_q = \beta_a^{1/4}, \qquad \delta_q = 1-(1-\delta_a)^{1/4}, \qquad \delta_{G,q} = 1-(1-\delta_{G,a})^{1/4}.$$

All other dimensionless parameters (e.g., $\nu, \phi, \alpha_j, \sigma, \mu_j, \psi_j$) are invariant to this conversion.

Consistency of steady-state ratios across frequencies The government block implies $K^G = B/\delta_G$ in steady state. In quarterly units, we preserve the same public-capital stock ratio by

[5] FRED series IDs are reported in the note to Table 1.

setting the per-quarter issuance/investment ratio to satisfy:

$$\left(\frac{B}{Y}\right)_q = \delta_{G,q}\overline{\left(\frac{K^G}{Y}\right)}.$$

Under small depreciation, this implies $4(B/Y)_q \approx (B/Y)_a$; in our calibration the approximation is numerically tight. This convention ensures that the quarterly IRFs are anchored to the same empirical steady-state stock target $\overline{K^G/Y}$ that disciplines the annual calibration.

4.3 Calibration

Table 1 reports the baseline parameterization. Parameters are divided into three groups: i) standard preference/technology parameters taken from the macro literature; ii) parameters pinned down by direct steady-state restrictions; and iii) parameters chosen to match labor-share levels and to discipline the productivity elasticity of public capital.

Table 1: Baseline calibration (U.S., annual)

Parameter	Value	Target / interpretation
Discount factor β	0.96	Standard annual calibration
Private depreciation δ	0.08	Standard annual calibration
Public depreciation δ_G	0.0426	Mean CFC^G/K^G
Debt-financed investment B/Y	0.0313	Match $\overline{K^G/Y}$ via $K^G = B/\delta_G$
Cross-sector elasticity ν	1.50	Substitution across sectoral composites
CES weight ϕ	0.442	Match mean non-services share $\bar{s}_c$
Capital share (capital-intensive) α_c	0.39	Higher capital intensity in sector c
Capital share (labor-intensive) α_ℓ	0.20	Lower capital intensity in sector ℓ
Within-sector elasticity σ	1.25	Substitution between K and L
Markup (capital-intensive) μ_c	1.11	Match mean aggregate labor share
Markup (labor-intensive) μ_ℓ	1.05	Lower markup in labor-intensive sector
Public exposure (capital-intensive) ψ_c	0.10	Higher infrastructure sensitivity
Public exposure (labor-intensive) ψ_ℓ	0.06	Lower infrastructure sensitivity

Note: All macro series are downloaded from FRED. Underlying sources are: BEA NIPA (GDP, services GDP, gross government investment, and government consumption of fixed capital), BEA Fixed Assets (government net stock of fixed assets), and Penn World Table (labor share). FRED series IDs: GDPA, A341RC1A027NBEA, A782RC1A027NBEA, A264RC1A027NBEA, K1GTOTL1ES000, LABSHPUSA156NRUG. Sample: 1970–2019. Frequency note: Calibration targets are computed from annual data and parameters are reported in annual units; for impulse-response analysis the model is solved at quarterly frequency using standard compounding conversions $\beta_q = \beta_a^{1/4}$, $\delta_q = 1-(1-\delta_a)^{1/4}$, and $\delta_{G,q} = 1-(1-\delta_{G,a})^{1/4}$.

Preferences and private-capital depreciation We set the discount factor at $\beta = 0.96$ (annual) and the private depreciation rate at $\delta = 0.08$ (annual), which implies the steady-state rental rate $r = \beta^{-1} - 1 + \delta$ (Equation (6)).

Public capital and debt-financed investment The public-capital depreciation rate δ_G is set to the sample mean $\bar{\delta}_G$. In steady state, the model implies $K^G = B/\delta_G$ (Equation (10)), so the steady-state issuance/investment ratio B/Y is pinned down by the empirical stock ratio:

$$\frac{B}{Y} = \delta_G \cdot \overline{\left(\frac{K^G}{Y}\right)} = 0.0313.$$

This choice ensures that the model matches the level of the public-capital stock in the data.

Final-good aggregation and sector weights We set the cross-sector elasticity $\nu = 1.5$, consistent with a substitution elasticity above unity. The CES weight ϕ is chosen to match $\bar{s}_c$ in the normalized steady state (i.e., $\phi = \bar{s}_c$). Sectoral productivity shifters (Z_c, Z_ℓ) are normalized so that $P_c = P_\ell = 1$ at the calibration point (Equation (13) combined with the sectoral pricing condition).

Within-sector technology and markups The capital share parameters satisfy $\alpha_c > \alpha_\ell$ and are chosen to match the level of the aggregate labor share. The within-sector elasticity is set to $\sigma = 1.25$ (common across sectors). Markups are set to $\mu_c = 1.11$ and $\mu_\ell = 1.05$, implying elasticities of substitution across varieties $\theta_c = \mu_c/(\mu_c - 1) \approx 10.1$ and $\theta_\ell = \mu_\ell/(\mu_\ell - 1) \approx 21.0$.

Public-capital exposure The exposure parameters satisfy $\psi_c > \psi_\ell$. We set $(\psi_c, \psi_\ell) = (0.10, 0.06)$, implying an average output elasticity of public capital around 0.08 at the calibrated sector share.

4.4 Simulation exercises

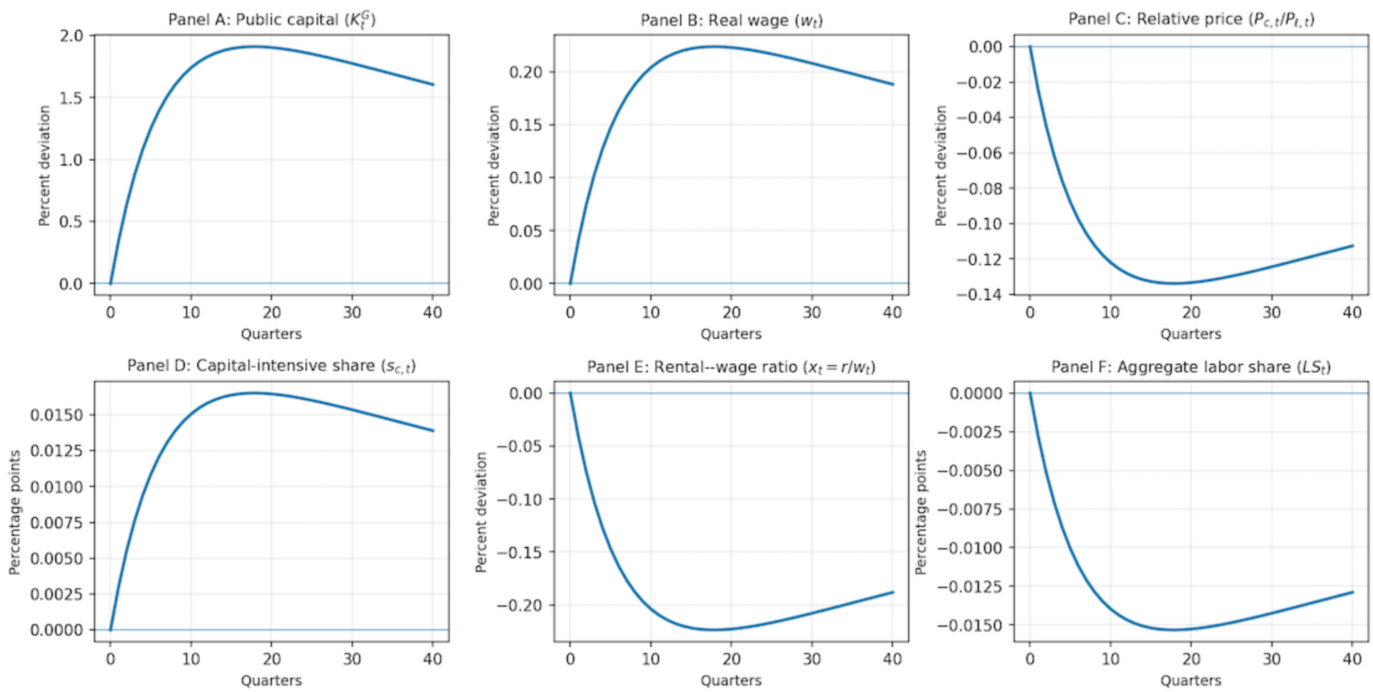
We simulate quarterly impulse responses that study the model's internal propagation under debt-financed public investment shocks.

Figure 1 reports impulse responses to a debt-financed public investment shock, implemented as an exogenous AR(1) increase in issuance B_t that maps one-for-one into public investment $I_t^G = B_t$, with public capital evolving as $K_{t+1}^G = (1 - \delta_G)K_t^G + B_t$.[6] All responses

[6] In the simulation underlying the figure, the impact shock is normalized to +0.25% of quarterly GDP (equivalently +1% of annual GDP) and the persistence is $\rho_B = 0.85$. The qualitative responses are robust to alternative normalizations.

are reported as deviations from the steady state: percent deviations for levels and prices, and percentage-point deviations for shares.

Figure 1: Impulse responses to a debt-finance public investment stock



Panel A of Figure 1 shows that the response of public capital is hump-shaped. This reflects the stock-flow structure: a persistent increase in B_t raises the flow of public investment, which accumulates gradually into K_t^G due to partial depreciation. As the shock mean-reverts, depreciation dominates and K_t^G slowly returns to steady state. Hence the peak occurs several quarters after impact rather than contemporaneously.

Panel B of Figure 1 illustrates that the real wage rises on impact and follows the same hump-shaped profile as K_t^G. The mechanism is that higher public capital increases sectoral effective productivity via $g_{j,t} = (K_t^G/\bar{K}^G)^{\psi_j}$, lowering unit costs for given factor prices. Under the numeraire normalization $P_t = 1$, equilibrium requires factor prices to adjust so that the aggregate price index remains fixed. The model delivers an increase in the real wage, consistent with higher marginal productivity of labor when public capital augments production.

Panel C reveals that the relative price of the capital-intensive composite declines. Because the capital-intensive sector has a higher exposure to public capital ($\psi_c > \psi_\ell$), the productivity boost from rising K_t^G is larger in sector c than in sector ℓ. This reduces the marginal cost and therefore the price of the capital-intensive composite relative to the labor-intensive one, generating a negative response of $P_{c,t}/P_{\ell,t}$.

Panel D shows that the revenue share of the capital-intensive sector increases. Given the CES structure of the final-good aggregator, relative demand responds to relative prices. With $\nu > 1$, the two sectoral composites are gross substitutes, so the decline in $P_{c,t}/P_{\ell,t}$ shifts expenditure toward sector c. In the model, this effect is captured by the revenue-share formula $s_{c,t} \propto \phi P_{c,t}^{1-\nu}$, implying that a fall in $P_{c,t}$ raises $s_{c,t}$ holding the numeraire fixed. The reallocation peaks when the relative price gap is largest and then unwinds gradually.

Panel D of Figure 1 illustrates that the rental–wage ratio declines. In the simulation, the rental rate r is anchored by the steady-state restriction $r = \beta^{-1} - 1 + \delta$, while w_t increases as discussed above. As a result, $x_t = r/w_t$ falls on impact and remains below steady state as long as wages stay elevated. This movement in x_t is the central driver of the within-sector labor-share adjustment because it changes the cost-minimizing factor shares implied by the CES technology.

Panel D of Figure 1 highlights that the aggregate labor share falls modestly and exhibits a hump-shaped decline, reaching its trough around the horizon at which the public-capital stock and the sectoral reallocation are strongest. Two complementary mechanisms generate this result. First, within-sector channel. With $\sigma > 1$, a decline in the relative price of capital services ($x_t \downarrow$) reduces the labor cost share in the sectoral unit-cost function, $\lambda_{j,t} = \left[1 + \left(\frac{\alpha_j}{1-\alpha_j}\right)^{\sigma} x_t^{1-\sigma}\right]^{-1}$. Since sectoral labor shares in revenue are $LS_{j,t} = \lambda_{j,t}/\mu_j$, the within-sector labor shares fall in both sectors as x_t declines. Second, between-sector channel. Panel D shows that expenditure shifts toward the capital-intensive sector. Because sector c is both more capital intensive ($\alpha_c > \alpha_\ell$) and features a higher markup ($\mu_c > \mu_\ell$), its sectoral labor share is lower than that of the labor-intensive sector. The increase in $s_{c,t}$ therefore mechanically lowers the aggregate labor share $LS_t = s_{c,t} LS_{c,t} + (1 - s_{c,t}) LS_{\ell,t}$.

Overall, the IRFs highlight the paper's key propagation mechanism: debt-financed public investment raises the public-capital stock and real wages, but—through relative-price changes and sectoral reallocation toward the capital-intensive, higher-markup sector—induces a temporary decline in the aggregate labor share.

5 Conclusion

Public infrastructure is often sold as a growth policy. It can raise productivity, lift wages, and improve the efficiency of private production. This paper argues that these aggregate gains do not, by themselves, imply a larger labor share. When public capital is financed by debt and it affects sectors asymmetrically, it can raise real wages and still push the economy toward activities and market structures that deliver a lower labor share.

The mechanism is simple and transparent. A local government issues one-period debt and uses the proceeds to finance public investment, so debt issuance maps directly into the public capital stock. Public capital enters production as a productivity shifter that can differ across sectors. In the two-sector environment, public capital affects the aggregate labor share through two distinct channels that are easy to confound in one-sector models. First, there is a within-sector factor-price channel: as public capital raises productivity, the equilibrium real wage adjusts, which changes the rental–wage ratio and therefore the cost-minimizing labor share inside each sector. Second, there is a between-sector market-share channel: if public capital reduces the relative price of the capital-intensive composite more strongly, expenditure shifts toward that sector when the two composites are substitutes. When the expanding sector is more capital intensive and/or has a higher markup, it carries a lower labor share, so the reallocation mechanically lowers the aggregate labor share.

Our main theoretical result provides an exact additive decomposition of the aggregate labor-share response into these two components. The decomposition clarifies which primitives govern each channel: substitution in production determines the sign and strength of the within-sector response, while substitution in demand and differences in public-capital exposure determine the sign and strength of the market-share response. Under empirically plausible restrictions—goods are substitutes across sectors, capital and labor are substitutes within sectors, the capital-intensive sector has lower labor share, and public capital is relatively more productive in that sector—both channels can be negative, so debt-financed public capital reduces the labor share in steady state.

The quantitative exercise illustrates the logic in a calibrated setting. We discipline the model with U.S. annual data and then study impulse responses at quarterly frequency using standard compounding conversions. A temporary increase in debt-financed public investment produces a hump-shaped response of the public-capital stock. Real wages rise, the relative price of the capital-intensive composite falls, and revenue shifts toward the capital-intensive sector. Consistent with the decomposition, these movements combine to generate a modest but systematic decline in the aggregate labor share during the transition.

The main policy lesson is not that public investment is undesirable. Rather, it is that the distributional consequences of productive public capital are state- and composition-dependent, and they should be evaluated alongside output and welfare.

In particular, first, build distributional accounting into infrastructure appraisal. Cost–benefit analysis of infrastructure projects is typically framed around output, congestion, or average productivity. Our results suggest adding a distributional layer: when public capital tilts relative prices and market shares, it can change labor's share even if wages rise. A practical implication is to track sectoral exposure to the project, the expected sectoral shift in revenue shares, and the implied change in the wage bill relative to profits.

Second, design investment toward "labor-complementary" public capital when distribution matters. If the policy objective includes supporting the labor share, the composition of public capital matters. Projects that disproportionately raise productivity in highly capital-intensive, high-markup activities are more likely to trigger the reallocation channel. In contrast, projects that raise effective productivity in labor-intensive activities (or lower markups via better market access and contestability) are more likely to distribute gains broadly. In practice, this points to prioritizing investments that ease bottlenecks faced by labor-intensive sectors (e.g., local transport connections, skills-related infrastructure, digital public services, and broad-based logistics that improve entry and competition) rather than projects whose benefits are concentrated in a narrow set of capital-heavy incumbents.

Third, pair debt-financed public investment with policies that recycle rents and support labor income. The mechanism in this paper operates partly through markups and sectoral reallocation. This suggests two complementary policy responses. First, competition and procurement policies that limit rent extraction can strengthen the pass-through of productivity gains into wages. Second, if the transition is expected to reduce the labor share, targeted fiscal instruments can offset distributional effects without undoing productive investment: for example, temporary earned-income tax credits, wage subsidies in exposed regions/sectors, or transfers financed by taxing windfall profits in sectors that expand and earn rents. The broader point is that the financing and redistribution package should be designed jointly with the investment program, not treated as an afterthought.

Last, plan for the transition, not only the long run. Even when the long-run gains from public capital are positive, the transition can involve a temporary decline in the labor share. Governments implementing large infrastructure pushes should anticipate and manage these transitional dynamics. Clear communication and automatic stabilizers that respond to distributional indicators can reduce political economy risks and increase the durability of investment plans.

The framework is intentionally stylized. Several extensions are natural and would sharpen the empirical and policy content: allowing for distortionary taxation or a richer fiscal menu; adding heterogeneous households and incomplete markets to quantify welfare trade-offs; introducing endogenous markups or entry to study how public capital affects market power; and embedding the mechanism in an open-economy setting in which relative prices interact with trade and the exchange rate. On the empirical side, the model delivers testable predictions: public investment episodes should be associated with sectoral reallocation toward more exposed sectors and a labor-share response that is stronger where exposure differentials and markups are larger. We view these as promising directions for future research.

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Appendix

A Model notation and parameter roles

Table A1 summarizes the full set of primitives and key endogenous objects used in the paper. This table is intended to: i) clarify economic interpretation, ii) document units/frequency, and iii) provide a single reference for the reader.

B Derivations and auxiliary results

This Appendix collects derivations that are either used repeatedly in the main text or are useful for replication. Where proofs are stated in the main text, the Appendix provides a consolidated reference.

B.1 Cost minimization in the intermediate sector

Fix (y, z, g) and consider:

$$\min_{K,L\geq 0} rK + wL \quad \text{s.t.} \quad zg\Big[\alpha K^{\frac{\sigma-1}{\sigma}} + (1-\alpha)L^{\frac{\sigma-1}{\sigma}}\Big]^{\frac{\sigma}{\sigma-1}} \geq y. \tag{B.1}$$

The Lagrangian is:

$$\mathscr{L} = rK + wL + \xi\Big(y - zg\big[\alpha K^{\frac{\sigma-1}{\sigma}} + (1-\alpha)L^{\frac{\sigma-1}{\sigma}}\big]^{\frac{\sigma}{\sigma-1}}\Big).$$

FOCs (interior) are:

$$r = \xi zg\Big[\alpha K^{\frac{\sigma-1}{\sigma}} + (1-\alpha)L^{\frac{\sigma-1}{\sigma}}\Big]^{\frac{1}{\sigma-1}} \alpha K^{-\frac{1}{\sigma}}, \tag{B.2}$$

$$w = \xi zg\Big[\alpha K^{\frac{\sigma-1}{\sigma}} + (1-\alpha)L^{\frac{\sigma-1}{\sigma}}\Big]^{\frac{1}{\sigma-1}} (1-\alpha)L^{-\frac{1}{\sigma}}. \tag{B.3}$$

Divide (B.2) by (B.3) to obtain

$$\frac{r}{w} = \frac{\alpha}{1-\alpha}\left(\frac{K}{L}\right)^{-\frac{1}{\sigma}} \quad \Rightarrow \quad \frac{K}{L} = \left(\frac{\alpha}{1-\alpha}\right)^{\sigma}\left(\frac{w}{r}\right)^{\sigma}. \tag{B.4}$$

Using (B.4) in the production constraint (binding at optimum) yields minimized variable cost

$$rK + wL = \frac{c(r,w)}{zg}\, y,$$

Table A1: Summary of notation and parameters

Symbol	Object	Interpretation / where it enters	Units
β	discount factor	household Euler equations (4)–(6)	annual (Table 1)
δ	private depreciation	private capital law of motion (3) and steady state (6)	annual (Table 1)
δ_G	public depreciation	public capital law of motion (8) and steady state (10)	annual (Table 1)
B_{t+1}	government debt issuance	policy rule $I_t^G = B_{t+1}$ in (9); maps to $\{K_t^G\}$	model units
K_t^G	public capital stock	productivity shifter $g_{j,t}$ in (18) and relative price (51)	model units
$\bar{K}^G$	scaling constant	normalizes $g_{j,t} = (K_t^G/\bar{K}^G)^{\psi_j}$ in (18)	level
ν	cross-sector elasticity	CES aggregator (11) and price index (13)	dimensionless
ϕ	CES weight	sectoral demand and shares (12)–(14)	dimensionless
θ_j	within-sector variety elasticity	monopolistic competition aggregator (15)–(16)	dimensionless
μ_j	sectoral markup	$\mu_j = \theta_j/(\theta_j - 1)$ in (26); pricing and labor share	dimensionless
F_j	productivity distribution	i.i.d. draws $z_{j,t}(i) \sim F_j$; defines Z_j in (29)	distribution
Z_j	CES productivity aggregator	$Z_j = \left(\int z^{\theta_j - 1} dF_j\right)^{1/(\theta_j - 1)}$ in (29)	level
α_j	CES share parameter	capital intensity in production (19) and unit cost (20)	dimensionless
σ_j	within-sector substitution	CES in (19) and unit cost (20); implies λ_j in (37)	dimensionless
ψ_j	public-capital exposure	sectoral exposure in $g_{j,t}$ (18); assumed $\psi_c > \psi_\ell$	dimensionless
r_t	rental rate of private capital	household Euler (4); steady state (6)	gross/real
w_t	real wage	household budget (2); wage feedback (68)	real
x_t	rental–wage ratio	$x_t = r_t/w_t$; drives λ_j in (37) and LS in (41)	dimensionless
$\lambda_{j,t}$	labor share in unit cost	defined in (36); closed form in (37); key for LS	dimensionless
$LS_{j,t}$	sectoral labor share (revenue)	$LS_{j,t} = \lambda_{j,t}/\mu_j$ (41)	dimensionless
LS_t	aggregate labor share	$LS_t = s_{c,t} LS_{c,t} + (1 - s_{c,t}) LS_{\ell,t}$ in (35)	dimensionless
$s_{c,t}$	sector c revenue share	definition (14); response (52) and decomposition (87)	dimensionless
R_t	gross return on government debt	household Euler (5) and steady state (6)	gross/real

Note. Parameters are reported in annual units in Table 1. For quarterly IRFs, we convert $(\beta, \delta, \delta_G)$ via exact compounding as described in subsection 4.2.

with $c(r,w)$ equal to the unit-cost function in Equation (20). This reproduces Lemma 1.

B.2 Labor share in variable cost and in revenue

From Equation (20), define $D \equiv \alpha^\sigma r^{1-\sigma} + (1-\alpha)^\sigma w^{1-\sigma}$ and $c(r,w) = D^{1/(1-\sigma)}$. Then the labor cost share in unit cost is:

$$\lambda(r,w) \equiv \frac{w\,\partial c(r,w)/\partial w}{c(r,w)} = \frac{(1-\alpha)^\sigma w^{1-\sigma}}{\alpha^\sigma r^{1-\sigma} + (1-\alpha)^\sigma w^{1-\sigma}} = \left[1 + \left(\frac{\alpha}{1-\alpha}\right)^\sigma \left(\frac{r}{w}\right)^{1-\sigma}\right]^{-1}, \tag{B.5}$$

which is Lemma 4. With constant markup pricing $p = \mu MC$, variable cost equals $(1/\mu)$ times revenue, so the revenue labor share satisfies $LS = \lambda/\mu$, yielding Proposition 1.

B.3 Differentiating $\lambda(x)$

Let $A \equiv \left(\frac{\alpha}{1-\alpha}\right)^\sigma$ and $\lambda(x) = [1 + Ax^{1-\sigma}]^{-1}$. Then

$$\frac{d\lambda}{dx} = -(1+Ax^{1-\sigma})^{-2} A(1-\sigma)x^{-\sigma}, \qquad \frac{d\lambda}{d\log x} = x\frac{d\lambda}{dx} = -(1-\sigma)\frac{Ax^{1-\sigma}}{(1+Ax^{1-\sigma})^2}.$$

Since $\lambda = \frac{1}{1+Ax^{1-\sigma}}$ and $1-\lambda = \frac{Ax^{1-\sigma}}{1+Ax^{1-\sigma}}$, we obtain

$$\frac{d\lambda}{d\log x} = -(1-\sigma)\lambda(1-\lambda), \tag{B.6}$$

which is Equation (86).

B.4 Mapping from debt issuance to public capital

Under the policy rule (9), $I_t^G = B_{t+1}$ and $K_{t+1}^G = (1-\delta_G)K_t^G + B_{t+1}$. If $B_{t+1} = B$ is constant, the steady state satisfies:

$$K^G = (1-\delta_G)K^G + B \quad \Rightarrow \quad K^G = \frac{B}{\delta_G},$$

as in Equation (10). Hence steady-state comparative statics in $\log K^G$ translate one-for-one to comparative statics in $\log B$.